Optimal Supply against Fluctuating Demand

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1 Introduction

Contrary to the common sense in economy, the optimal supply does not always agree with the average demand. This was pointed out by Sornette et al. (1999), who analyzed a bakery model where a daily demand fluctuates. They derived the formula of the optimal supply, assuming the fluctuations obey a uniform distribution. Although their result is reasonable and meaningful, it is not clear how the result will change if we consider a different distribution.

In this note, we extend the model to general probability distributions. In particular, we calculate the optimal supply for Gaussian distribution, which is more realistic in a market.

2 Model and analysis of Sornette, Stauffer and Takayasu

In this section we review the model and analysis of Sornette et al. Let us consider a bakery shop where baked croissants are sold every day. A question is how many croissants should be baked a day to make the maximal profit. We define the variables as follows.

- \(x\) : the selling price of a croissant.
- \(y\) : its production cost.
- \(s\) : the production number of croissants per day (supply).
- \(n\) : the number of croissants requested by customers per day (demand).
- \(D\) : the average demand, i.e., \(D \equiv \langle n \rangle\).

The expectation of the total profit \(L(s)\) is given by

\[
L(s) \equiv \langle x \min(n, s) - y s \rangle = x \int_0^s nP(n)dn + x s \int_s^\infty P(n)dn - y s, \tag{1}
\]
where \( P(n) \) is the probability distribution of \( n \).

Sornette et al. assumed, for simplicity, a uniform distribution,

\[
P_u(n) = \begin{cases} 
\frac{1}{2\delta} & \text{for } D - \delta \leq n \leq D + \delta \\
0 & \text{for } n < D - \delta, \ D + \delta < n.
\end{cases}
\]  

(2)

Then one can integrate (1) as

\[
L(s) = -\frac{x}{4\delta} \left( s - D - \delta \left(1 - \frac{2y}{x}\right) \right)^2 + (x - y) \left(D - \frac{\delta y}{x}\right).
\]  

(3)

\( L(s) \) takes the maximum value when \( s \) takes

\[
s_{\text{max}} \equiv D + \delta \left(1 - \frac{2y}{x}\right).
\]  

(4)

This shows that, if the cost per price, \( y/x \), is larger (smaller) than half, the optimal demand, \( s_{\text{max}} \), is smaller (larger) than the average demand, \( D \).

3 Optimal supply for Gaussian distribution

Let us re-analyze (1) for general probability distributions. We do not have to integrate (1) directly, because what we want to know is the optimal supply \( s_{\text{max}} \), which is given by

\[
\frac{dL}{ds}(s_{\text{max}}) = x \int_{s_{\text{max}}}^{\infty} P(n)dn - y = 0.
\]  

(5)

This equation is very simple but quite general; it gives the optimal supply for general probability distributions. For arbitrary distribution function \( P(n) \), one can easily obtain the optimal supply \( s_{\text{max}} \) by integrating (5) numerically.

If we assume Gaussian distribution,

\[
P_G(n) \equiv \frac{1}{\sqrt{2\pi}\sigma} \exp \left[ -\frac{(n - D)^2}{2\sigma^2} \right],
\]  

(6)

we can integrate (5) analytically as follows. The integration in (5) is evaluated as

\[
\int_{s_{\text{max}}}^{\infty} P_G(n)dn = \frac{1}{2} - \frac{1}{2} \text{Erf} \left( \frac{s_{\text{max}} - D}{\sqrt{2}\sigma} \right),
\]  

(7)

where \( \text{Erf} \) is the error function, which is defined as

\[
\text{Erf} \ z \equiv \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt.
\]  

(8)
Then we arrive at the formula of the optimal supply for Gaussian distribution,

\[
  \frac{s_{\text{max}} - D}{\sigma} = \sqrt{2} \text{Erf}^{-1} \left( 1 - \frac{2y}{x} \right) \\
  = \sqrt{\frac{\pi}{2}} \left( 1 - \frac{2y}{x} \right) + \frac{\sqrt{2 \pi}}{24} \left( 1 - \frac{2y}{x} \right)^3 + O \left( \left( 1 - \frac{2y}{x} \right)^5 \right) \quad \text{(Gaussian).} \quad (9)
\]

Because the cost is usually in the range \(0.3x < y < 0.7x\), which reads \(|1 - 2y/x| < 0.4\), the first-order approximation in (9) is sufficient in most cases.

For reference, we rewrite the result for the uniform distribution (4). Because the variance of the uniform distribution (2) is evaluated as \(\sigma^2 = \frac{x^2}{3}\), (4) is rewritten as

\[
  \frac{s_{\text{max}} - D}{\sigma} = \sqrt{3} \left( 1 - \frac{2y}{x} \right) \quad \text{(uniform).} \quad (10)
\]

We see that the difference between \(\sqrt{\frac{\pi}{2}} \approx 1.25\) in (9) and \(\sqrt{3} \approx 1.73\) in (10) is not negligible. Contrary to the speculation of Sornette et al., however, the critical value of the cost-to-price ratio, \(y/x = 1/2\), is unchanged. Because Gaussian distribution is more realistic, our simple formula (9) is useful in a real market to earn the largest income on average.

References

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