Hardy-type inequalities for the generalized Mehler transform

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Abstract

We establish Hardy-type inequalities for the generalized Mehler transform on the real Hardy space $H^p$, $0 < p < 1$.

1. Introduction and Results

Let $0 < p \leq 1$ and $H^p(\mathbb{R})$ be the real Hardy space, that is, the space of the boundary distributions $f(x) = \Re F(x)$ of the real parts $\Re F(z)$ of functions $F(z)$ in the Hardy space $H^p(\mathbb{R}_+^2) = \{ F(z); \text{analytic in } \mathbb{R}_+^2 \text{ and } \| F \|_{H^p(\mathbb{R}_+^2)} = \sup_{t > 0} (\int_{-\infty}^{\infty} |F(x + it)|^p dx)^{1/p} < \infty \}$ on the upper half plane $\mathbb{R}_+^2 = \{ z = x + it; t > 0 \}$, with the norm $\| f \|_{H^p} = \| F \|_{H^p(\mathbb{R}_+^2)}$. Then, the Fourier transform $\hat{f}$ of $f \in H^p(\mathbb{R})$ is a continuous function and satisfies the inequality

$$\int_{-\infty}^{\infty} |\hat{f}(\xi)|^p |\xi|^{-2} d\xi \leq C \| f \|^p_{H^p},$$

which is well-known as Hardy’s inequality for $H^p(\mathbb{R})$ (cf. [7, Corollary 7.23], [21, p.128]).

The aim of this paper is to establish an analogue of this inequality for the generalized Mehler transform.

The generalized Mehler transform is defined as follows. Let $m$ be a real number such that $m \leq 1/2$, and define

$$K^m(x, y) = k_m(x)(\sinh y)^{1/2} P_{-1/2 + ix}^m(\cosh y),$$

where

$$k_m(x) = \left| \frac{\Gamma(1/2 - m - ix)}{\Gamma(-ix)} \right|,$$

and $P_{-1/2 + ix}^m(z)$ is the Legendre function of order $m$ and degree $-1/2 + ix$, which is given by using the hypergeometric function as follows:

$$P_{-1/2+ix}^m(z) = \frac{1}{\Gamma(1-m)} \left( \frac{z+1}{z-1} \right)^{m/2} F(1/2 - ix, 1/2 + ix; 1 - m; 1/2 - z/2).$$

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The following transforms

\[ G^m(f; y) = \int_0^\infty f(x)K^m(x, y) \, dx, \]

\[ H^m(g; x) = \int_0^\infty g(y)K^m(x, y) \, dy. \]

are called the generalized Mehler transform. We remark that if \( f, g \in L^1[0, \infty) \), then the values \( G^m(f; y), H^m(g; x) \) exist for every \( x, y > 0 \) since \( |K^m(x, y)| \leq C, x > 0, y > 0, m \leq 1/2 \) (cf. [20]). Let us call \( G^m \) and \( H^m \) the G-type transform of order \( m \) and the H-type transform of order \( m \), respectively. It is known that \( K^{1/2}(x, y) = \sqrt{2/\pi} \cos xy \) and \( K^{-1/2}(x, y) = \sqrt{2/\pi} \sin xy \). Thus the H-type and G-type transforms of order \( 1/2 \) are the cosine transform and, those transforms of order \(-1/2\) are the sine transform. The above classical Hardy inequality leads to the following inequalities

\[
\int_0^\infty |G^{\pm 1/2}(f, y)|^p y^{p-2} \, dy \leq C\|f\|_{H^p(\mathbb{R})}^p,
\]

\[
\int_0^\infty |H^{\pm 1/2}(f, y)|^p y^{p-2} \, dy \leq C\|f\|_{H^p}^p,
\]

where \( f \in H^p(\mathbb{R}) \) with \( \text{supp } f \subset [0, \infty) \) and \( 0 < p \leq 1 \).

In this paper, we shall investigate Hardy-type inequalities for the G-type and H-type transforms of arbitrary order \( m < 1/2 \) on the space

\[ H^p[0, \infty) = \{ f \in H^p(\mathbb{R}) : \text{supp } f \subset [0, \infty) \}, \quad 0 < p \leq 1, \]

and obtain the following:

**Theorem 1.** (i) Let \(-m + 1/2 > 0\) and \(0 < p \leq 1\). Then, there exists a constant \( C \) such that

\[
\int_1^\infty |G^m(f; y)|^p y^{p-2} \, dy \leq C\|f\|_{H^p[0, \infty)}, \quad f \in H^p[0, \infty).
\]

(ii) Let \(-m + 1/2 > 0\) and \(0 < p \leq 1\). Suppose that \([1/p] \leq [-m + 1/2]\). Then, there exists a constant \( C \) such that

\[
\int_0^1 |G^m(f; y)|^p y^{p-2} \, dy \leq C\|f\|_{H^p[0, \infty)}, \quad f \in H^p[0, \infty).
\]

**Theorem 2.** (i) Let \(-m + 1/2 > 0\) and \(0 < p \leq 1\). Suppose that \(1/p - 1 < -m + 1/2\). Then, there exists a constant \( C \) such that

\[
\int_1^\infty |H^m(g; x)|^p x^{p-2} \, dx \leq C\|g\|_{H^p[0, \infty)}, \quad g \in H^p[0, \infty).
\]

If \(-m + 1/2 = 1, 2, 3, \ldots\), then the above inequality holds for every \( p \) with \( 0 < p \leq 1 \).

(ii) Let \(-m + 1/2 > 0\) and \(1/2 < p \leq 1\). Suppose that \(1/p - 1 < -m + 1/2\). Then, there exists a constant \( C \) such that

\[
\int_0^1 |H^m(g; x)|^p x^{p-2} \, dx \leq C\|g\|_{H^p[0, \infty)}, \quad g \in H^p[0, \infty).
\]
**Corollary 1.** Let $1/2 < p \leq 1$ and $-m + 1/2 = 1, 2, 3, \ldots$. Then, there exist constants $C$ such that

$$
\int_0^\infty |G_m(f; y)|^p y^{p-2} dy \leq C \|f\|_{H^p[0, \infty)}, \quad f \in H^p[0, \infty),
$$

and

$$
\int_0^\infty |H_m(g; x)|^p x^{p-2} dy \leq C \|g\|_{H^p[0, \infty)}, \quad g \in H^p[0, \infty).
$$

There are several results related to Hardy’s inequality. A Hardy-type inequality for the Hankel transform is in [11], and the inequalities for Hermite and Laguerre expansions are in [10] and [12]. Hardy’s inequality associated with the $n-1$ dimensional unit sphere in $\mathbb{R}^n, n \geq 3$ is in [4], and the ones for higher-dimensional Hermite and special Hermite expansions are in [18]. Some other inequalities of Hardy-type will be found in Colzani and Travaglini [5], Thangavelu [22], Betancor and Rodríguez-Mesa [2], Guadalupe and Kolyada [8], Kanjin and Sato [13], Sato [19], Balasubramanian and Radha [1].

We give some facts about the generalized Mehler transform. The usual generalized Mehler transform pair is the following:

$$
g(u) = \int_0^\infty f(x) P_{m-1/2+ix}^m(u) \, dx,
$$

$$
f(x) = \pi^{-1} x \sinh \pi x \Gamma(1/2 - m + ix) \Gamma(1/2 - m - ix) \int_1^\infty g(u) P_{m-1/2+ix}^m(u) \, dx.
$$

Conditions for the inversion of this pair will be found, for example, in [15]. According to [20], we reformulate this pair. We note that

$$
k_m^2(x) = \pi^{-1} x \sinh \pi x \Gamma(1/2 - m + ix) \Gamma(1/2 - m - ix),
$$

and then we have

$$
g(\cosh y)(\sinh y)^{1/2} = \int_0^\infty f(x) k_m(x) K^m(x, y) \, dx,
$$

$$
\frac{f(x)}{k_m(x)} = \int_0^\infty g(\cosh y)(\sinh y)^{1/2} K^m(x, y) \, dy.
$$

Rewriting $g(\cosh y)(\sinh y)^{1/2}$ and $f(x)/k_m(x)$ with $g(y)$ and $f(x)$, again, we have H-type and G-type transforms.

The generalized Mehler transform is a special case of the Jacobi transform. We follow the notations of Koornwinder [14]. Let $\phi_x^{(\alpha, \beta)}(t)$ be the Jacobi functions:

$$
\phi_x^{(\alpha, \beta)}(t) = F((\alpha + \beta + 1 - i\lambda)/2, (\alpha + \beta + 1 + i\lambda)/2; \alpha + 1; \sinh^2 t).
$$

Put

$$
\Delta_{\alpha, \beta}(t) = (2 \sinh t)^{2\alpha+1}(2 \cosh t)^{2\beta+1}.
$$

The Jacobi transform of a function $f$ is defined by

$$
\hat{f}(\lambda) = \int_0^\infty f(t) \phi_x^{(\alpha, \beta)}(t) \Delta_{\alpha, \beta}(t) \, dt.
$$

Let $G$ be a connected noncompact semisimple Lie group with finite center, and fix a maximal compact subgroup $K$. Associated to $G$ there are constants $p, q =$
0, 1, 2, . . . determined by the geometry of the symmetric space \( G/K \) such that \( n = \dim(G/K) = p + q + 1 \). Let

\[
\alpha = \frac{p + q - 1}{2}, \quad \beta = \frac{q - 1}{2},
\]

that is,

\[
p = 2(\alpha - \beta), \quad q = 2\beta + 1, \quad n = 2\alpha + 2.
\]

Then the Jacobi functions \( \phi^{(\alpha, \beta)}_\lambda(t) \) and the Jacobi transform appear as the spherical functions and the spherical transform on \( G/K \). The Plancherel theorem for the Jacobi transform is as follows:

\[
\int_0^\infty |f(t)|^2 \Delta_{\alpha, \beta}(t) \, dt = \frac{1}{2\pi} \int_0^\infty |\hat{f}(\lambda)|^2 |c(\lambda)|^{-2} \, d\lambda
\]

if \( \alpha > -1 \) and \( \alpha \pm \beta + 1 \geq 0 \). Here,

\[
c(\lambda) = \frac{2^{\rho - i\lambda}(\alpha + 1)\Gamma(i\lambda)}{\Gamma((i\lambda + \rho)/2)\Gamma((i\lambda + \alpha - \beta + 1)/2)}, \quad \rho = \alpha + \beta + 1.
\]

There are relations between the generalized Mehler transform and the Jacobi transform. Let

\[
\alpha = \beta = -m, \quad x = \lambda/2, \quad y = 2t.
\]

Then we have the following.

\[
\Delta_{\alpha, \beta}(t) = (2 \sinh y)^{-2m+1}, \quad \phi^{(\alpha, \beta)}_\lambda(t) = 2^{-m}(\sinh y)^{-m}P^{-m-1/2+i\lambda}(\cosh y),
\]

\[
\hat{f}(\lambda) = \frac{2^{-2m}i\lambda}{k_m(x)} H^m(g; x), \quad g(y) = 2^{-m}(\sinh y)^{-m+1/2}f(y/2),
\]

\[
|c(\lambda)|^{-2} = \frac{2^{4m} \pi}{(\alpha + \beta + 1)^2 k_m^2(x)}.
\]

In this case, the Plancherel theorem is as follows: If \( m \leq 1/2 \), then

\[
\int_0^\infty |g(y)|^2 \, dy = \int_0^\infty |H^m(g; x)|^2 \, dx, \quad g \in L^2((0, \infty), dy),
\]

and

\[
\int_0^\infty |f(x)|^2 \, dx = \int_0^\infty |G^m(f; y)|^2 \, dy, \quad f \in L^2((0, \infty), dx).
\]

A main tool for the proof of the theorems is the atomic decomposition characterization of the real Hardy spaces. Let \( 0 < p \leq 1 \) and

\[
N = \left[1/p\right] - 1
\]

where the notation \([x]\) means that the greatest integer not exceeding \( x \). An \( H^p \) atom is a real valued function \( a(x) \) on \( \mathbb{R} \) so that (i) \( a(x) \) is supported in an interval \([c, c + h] \), (ii) \(|a(x)| \leq h^{-1/p} \) a.e. \( x \), and (iii) \( \int_{\mathbb{R}} a(x)x^k \, dx = 0 \) for all \( k = 0, 1, 2, \ldots, N \). The elements \( f \in H^p([0, \infty)) \) are characterized as follows: \( f \in H^p(\mathbb{R}) \) and \( \text{supp} \, f \subset [0, \infty) \) if and only if \( f = \sum_{j=0}^\infty \lambda_j a_j \), where every \( a_j \) is an \( H^p \) atom with \( \text{supp} \, a_j \subset [0, \infty) \) and \( \sum_{j=0}^\infty |\lambda_j|^p < \infty \). Moreover, the norm \( ||f||_{H^p([0, \infty))} \) is equivalent to \( \inf(\sum_{j=0}^\infty |\lambda_j|^p)^{1/p} \), the infimum being taken over all such decompositions, and the series \( \sum_{j=0}^\infty \lambda_j a_j \) converges in \( H^p \) norm, consequently, also in the sense of tempered distributions. For this characterization, we refer to [17].
The case \( p = 1 \) is in [7, III.7]. Related results are in [21, III.5.22], [3], [6], [9] and [16].

Because of the above characterization, we will be able to deduce the theorems from estimation of higher derivatives of the kernel \( K^m(x,y) \). The estimation will be stated in the following section, and the proof of the theorems will be give in the section 4.

2. Main Estimates

For the proof of the theorems, we need to know about asymptotic behavior of the higher order derivatives \( \frac{\partial^j K^m(x,y)}{\partial x^j} \) and \( \frac{\partial^j K^m(x,y)}{\partial y^j} \), \( j = 0, 1, 2, \ldots \) in variables \( x \) and \( y \). Schindler [20] has obtained precise asymptotic formulas of \( K^m(x,y) \) and the first order derivatives \( \frac{\partial K^m(x,y)}{\partial x} \) and \( \frac{\partial K^m(x,y)}{\partial y} \). These formulas are enough to obtain our theorems in the case \( p = 1 \). We would like to consider Hardy-type inequalities for all \( p \) with \( 0 < p \leq 1 \). This forces us to estimate the higher order derivatives. Our main estimates are the following Lemma 1 and Lemma 2 in which the letter \( C \) means positive constants independent of \( x \) and \( y \) not necessarily the same at each occurrence.

**Lemma 1.** Let \( -m + 1/2 > 0 \), and put \( M = [-m + 1/2] \). Then the following inequalities hold:

For \( 0 < x < 1 \), \( 0 < y < 1 \):

\[
\left| \frac{\partial^j}{\partial x^j} K^m(x,y) \right| \leq C y^{-m+1/2}, \quad j = 0, 1, 2, \ldots.
\]

For \( 0 < x < 1 \), \( 1 \leq y \):

\[
\left| \frac{\partial^j}{\partial x^j} K^m(x,y) \right| \leq C y^j, \quad j = 0, 1, 2, \ldots.
\]

For \( 1 \leq x \), \( 1 \leq y \):

\[
\left| \frac{\partial^j}{\partial x^j} K^m(x,y) \right| \leq C y^j, \quad j = 0, 1, 2, \ldots.
\]

For \( 1 \leq x \), \( 0 < y < 1 \):

\[
\left| \frac{\partial^j}{\partial x^j} K^m(x,y) \right| \leq C \left\{ \begin{array}{ll}
y^j, & j = 0, 1, 2, \ldots, M, \\
y^{-m+1/2}, & j = M + 1, \ldots.
\end{array} \right.
\]

**Lemma 2.** Let \( -m + 1/2 > 0 \), and put \( M = [-m + 1/2] \), \( \delta = -m + 1/2 - M \). Then the following inequalities hold:

For \( 0 < x < 1 \), \( 0 < y < 1 \):

\[
\left| \frac{\partial^j}{\partial y^j} K^m(x,y) \right| \leq C x, \quad j = 0, 1, 2, \ldots, M,
\]

\[
\left| \frac{\partial^M}{\partial y^M} K^m(x,y) - \frac{\partial^M}{\partial y^M} K^m(x,\xi) \right| \leq C x |y - \xi|^\delta, \quad 0 < \xi < 1.
\]

For \( 0 < x < 1 \), \( 1 \leq y \):

\[
\left| \frac{\partial^j}{\partial y^j} K^m(x,y) \right| \leq C x, \quad j = 1, 2, 3, \ldots.
\]
For $1 \leq x, 1 \leq y$:
\begin{equation}
\left| \frac{\partial^j}{\partial y^j} K^m(x, y) \right| \leq C x^j, \quad j = 0, 1, 2, \ldots
\end{equation}

For $1 \leq x, 0 < y < 1$:
\begin{equation}
K^m(x, y) = \tilde{k}_m(x)(xy)^{1/2} J_{-m} (xy) + E_m(x, y),
\end{equation}
with \( \left| \tilde{k}_m(x) \right| \leq C \), \( \left| \frac{\partial^j}{\partial y^j} E_m(x, y) \right| \leq C x^j, \quad 0 \leq j < -m + 3/2, \)

and if \(-m + 1/2 = 1, 2, 3, \ldots\), then
\begin{equation}
\left| \frac{\partial^j}{\partial y^j} K^m(x, y) \right| \leq C x^j, \quad j = 0, 1, 2, \ldots
\end{equation}

The above estimates are obtained by reexamining and refining the arguments that Schindler [20] used to get the asymptotic formulas for \( K^m(x, y), \partial K^m(x, y)/\partial x \) and \( \partial K^m(x, y)/\partial y \). The work is routine, but a little hard. The details are omitted in this paper.

3. The Generalized Mehler Transform for $H^p$ with $0 < p \leq 1$

Let $0 < p \leq 1$ and $-m + 1/2 > 0$. We shall discuss defining the transforms \( \mathcal{G}^m(f; y) \) and \( \mathcal{H}^m(f; x) \) of \( f \in H^p[0, \infty) \). We use the fact that an element of the Lipschitz space \( \Lambda_{1/p-1}(\mathbb{R}) \) defines a continuous linear functional of \( H^p(\mathbb{R}) \) (cf. [7, III.5]).

Fix $y > 0$. We take a function \( \kappa^m_y \) in $x$ such that
\( \kappa^m_y \in \Lambda_{1/p-1}(\mathbb{R}), \quad \kappa^m_y(x) = K^m(x, y), \quad x > 0, \)

and the transform \( \mathcal{G}^m(f; y) \) of \( f \in H^p[0, \infty) \) is defined by
\(\mathcal{G}^m(f; y) = \langle \kappa^m_y, f \rangle, \quad y > 0, \)

where the existence of such a function \( \kappa^m_y \) will be discussed below. Then for an atom \( a \in H^p[0, \infty) \), we have
\(\mathcal{G}^m(a; y) = \langle \kappa^m_y, a \rangle = \int_0^\infty a(x) K^m(x, y) \, dx, \)

and for the atomic decomposition \( f = \sum_{j=0}^\infty \lambda_j a_j(x) \) of \( f \in H^p[0, \infty) \),
\(\mathcal{G}^m(f; y) = \sum_{j=0}^\infty \lambda_j \kappa^m_y(a_j; y). \)

We see that the transform \( \mathcal{G}^m(f; y) \) is independent of the choice of an extension \( \kappa^m_y \in \Lambda_{1/p-1}(\mathbb{R}) \). In the same way, for fix \( x > 0 \), we take a function \( \kappa^m_x \) in $y$ such that
\( \kappa^m_x \in \Lambda_{1/p-1}(\mathbb{R}), \quad \kappa^m_x(y) = K^m(x, y), \quad y > 0, \)

and the transform \( \mathcal{H}^m(f; x) \) of \( f \in H^p[0, \infty) \) is defined by
\(\mathcal{H}^m(f; x) = \langle \kappa^m_x, f \rangle, \quad x > 0, \)

where we shall show that it is possible to take a function \( \kappa^m_x \). Then for an atom \( a \in H^p[0, \infty) \), we have
\(\mathcal{H}^m(a; x) = \langle \kappa^m_x, a \rangle = \int_0^\infty a(y) K^m(x, y) \, dy, \)
and for the atomic decomposition $f = \sum_{j=0}^{\infty} \lambda_j a_j(y)$ of $f \in H^p[0, \infty)$,
\[
\mathcal{H}^m(f; x) = \sum_{j=0}^{\infty} \lambda_j < \kappa_x^m, a_j = \sum_{j=0}^{\infty} \lambda_j \mathcal{H}^m(a_j; x).
\]
The transform $\mathcal{H}^m(f; y)$ is independent of the choice of an extension $\kappa_y^m \in \Lambda_1/p-1(\mathbb{R})$

Let us discuss the existence of extensions $\kappa_y^m$ and $\kappa_x^m$. Fix a positive $\eta$. We examine the kernel
\[
K^m(x, y) = k_m(x)(\sinh y)^{1/2} P_{-1/2+ix}^m(\cosh y)
\]
\[
= k_m(x) \frac{1}{\Gamma(1 - m)} \frac{(\cosh y + 1)^m}{(\sinh y)^{m-1/2}} \cdot F(1/2 - ix, 1/2 + ix; 1 - m; (1 - \cosh y)/2)
\]
as a function in $x$. We note here that for fixed $z$ in the plane $\mathbb{C}$ cut along $[1, \infty]$, the hyper geometric function $F(\alpha, \beta; \gamma; z)$ is an entire function of $\alpha$ and $\beta$, and a meromorphic function of $\gamma$, with simple poles at the points $\gamma = 0, -1, -2, \ldots$. Thus we see that the function $(\sinh y)^{1/2} P_{-1/2+ix}^m(\cosh y)$ is an entire function in $x$. The function $\mathcal{H}^m(x, y)$ satisfies
\[
k_m(x) = \frac{(1 - iy)(-iy)\Gamma(1/2 - m - iy)}{\Gamma(2 - iy)} = \frac{|(1 - iy)(-iy)|}{\Gamma(2 - iy)} \left| \frac{\Gamma(1/2 - m - iy)}{\Gamma(2 - iy)} \right|, \quad x > 0.
\]
Since $\Gamma(1/2 - m - iy)/\Gamma(2 - iy)$ is a holomorphic function with no zeros in $|x| < 3/2$, it follows that $|\Gamma(1/2 - m - iy)/\Gamma(2 - iy)| \in C^\infty(-3/2, 3/2)$. By these considerations, we can take $\kappa_y^m \in C^\infty(\mathbb{R})$ such that
\[
\kappa_y^m(x) = \begin{cases}
K^m(x, y), & x > 0, \\
0, & x < -\eta,
\end{cases}
\]
where $\eta$ is a positive constant. By Lemma 1, we see that $\kappa_y^m \in \Lambda_\rho(\mathbb{R})$ for every $\rho > 0$.

Fix a positive $x$. By the properties of the hyper geometric functions, we see that there exists a function $h_x(y) \in C^\infty(\mathbb{R})$ such that
\[(\sinh y)^{1/2} P_{-1/2+ix}^m(\cosh y) = (\sinh y)^{-m+1/2} h_x(y), \quad y > 0,
\]
and then for a positive constant $\eta > 0$ there exists a function $p_2^m$ such that
\[
p_2^m(y) = \begin{cases}
(\sinh y)^{-m+1/2} h_x(y), & y > -\eta, \\
0, & y \leq -2\eta,
\end{cases}
\]
and $p_2^m \in C^\infty(\mathbb{R} \setminus \{0\})$ if $-m + 1/2 \neq 0, 1, 2, \ldots$, and $p_2^m \in C^\infty(\mathbb{R})$ if $-m + 1/2 = 0, 1, 2, \ldots$. By Lemma 2, we see that
\[
|\frac{\partial^j}{\partial y^j} (\sinh y)^{1/2} P_{-1/2+ix}^m(\cosh y)| \leq C_{j,m}(x), \quad y > 0, \quad j = 0, 1, 2, \ldots.
\]
Thus we have that for \(-m + 1/2 = 1, 2, 3, \ldots\),
\[
\left| \frac{\partial^j}{\partial y^j} p^m_x(y) \right| \leq C'_{j,m}(x), \quad -\infty < y < \infty, \quad j = 0, 1, 2, \ldots,
\]
and that \(\kappa^m_x \in \Lambda_\rho(\mathbb{R})\) for every \(\rho > 0\), where
\[
\kappa^m_x(y) = k_m(x)p^m_x(y), \quad -\infty < y < \infty.
\]
Here, \(C_{j,m}(x), C'_{j,m}(x)\) are constants independent of \(y\) and depending on \(m, j\) and \(x\). In the case \(-m + 1/2 \neq 1, 2, 3, \ldots\), we see that
\[
\left| \frac{\partial^j}{\partial y^j} p^m_x(y) \right| \leq \begin{cases} 
C_{j,m}(x), & -\eta < y < \eta, \quad j = 0, 1, 2, \ldots, M, \\
C''_{j,m}(x), & \eta \leq |y|, \quad j = 0, 1, 2, \ldots,
\end{cases}
\]
where \(M = [-m + 1/2]\). Put \(\delta = -m + 1/2 - M > 0\). Then it is easy to see that
\[
\left| \frac{\partial^M}{\partial y^M} p^m_x(y) - \frac{\partial^M}{\partial y^M} p^m_x(y') \right| \leq C|y - y'|^\delta, \quad y, y' \in (-\eta, \eta).
\]
The inequalities (12) and (13) lead to \(\kappa^m_x \in \Lambda_\rho(\mathbb{R})\) for every \(\rho\) with \(0 < \rho \leq -m + 1/2\).

Summarizing the above discussion, we have the following.

**Lemma 3.** (i) Let \(0 < p \leq 1\) and \(-m + 1/2 > 0\). Then, the \(G\)-transform \(G^m\) is well-defined on \(H^p[0, \infty)\).

(ii)\(\,\) Let \(0 < p \leq 1\) and suppose \(1/p - 1 \leq -m + 1/2\). Then, the \(H\)-transform \(H^m\) is well-defined on \(H^p[0, \infty)\).

(iii) Let \(-m + 1/2 = 0, 1, 2, \ldots\), then the \(H\)-transform \(H^m\) is well-defined on \(H^p[0, \infty)\) for every \(p\) with \(0 < p \leq 1\).

4. Proofs of Theorems

We shall turn to proofs of the theorems. Let \(f \in H^p[0, \infty), 0 < p \leq 1\). Then we have \(f = \sum_{j=0}^\infty \lambda_j a_j\), where every \(a_j\) is an \(H^p\) atom with \(\text{supp} a_j \subset [0, \infty)\) and \(\sum_{j=0}^\infty |\lambda_j|^p < \infty\). Moreover, the norm \(\|f\|_{H^p[0, \infty)}\) is equivalent to \(\inf(\sum_{j=0}^\infty |\lambda_j|^p)^{1/p}\), the infimum being taken over all such decompositions. Because of the decomposition, to prove the theorems it is enough to show that for \(H^p\)-atoms \(a\) with \(\text{supp} a \subset [0, \infty)\),
\[
\int_A |G^m(a; y)|^p y^{-p-2} dy \leq C_1, \quad \int_A |H^m(a; x)|^p x^{-p-2} dx \leq C_2
\]
with constants \(C_1\) and \(C_2\) independent of \(a\) under the conditions we need for \(p\) and \(m\), where \((A, B) = (0, 1)\) or \((A, B) = (1, \infty)\). For the continuity of the transforms leads to
\[
G^m(f; y) = \sum_{j=0}^\infty \lambda_j G^m(a_j; y), \quad H^m(f; x) = \sum_{j=0}^\infty \lambda_j H^m(a_j; x),
\]
and if (14) holds, then we have that
\[
\int_A^B |G^m(f; y)|^p y^{p-2} \, dy \leq \sum_{j=0}^{\infty} |\lambda_j|^p \int_A^B |G^m(a_j; y)|^p y^{p-2} \, dy
\]
\[
\leq C_1 \sum_{j=0}^{\infty} |\lambda_j|^p \leq C'_1 \|f\|_{H^p}^p,
\]
and \( \int_A^B |H^m(f; y)|^p y^{p-2} \, dy \leq C'_2 \|f\|_{H^p}^p \), where \( C'_1 \) and \( C'_2 \) are constants independent of \( f \in H^p[0, \infty) \).

**Proof of Theorem 1 (i).** Let \( 0 < p \leq 1 \) and \( -m + 1/2 > 0 \). Let \( a \) be an \( H^p \)-atom with the support interval \( [c' - h, c'] \subset [0, \infty) \). We put \( N = [1/p] - 1 \). The vanishing mean property of atoms leads to
\[
|G^m(a; y)| \leq \int_{c' - h}^{c'} |a(x)| \left| \frac{\partial^{N+1}}{\partial x^{N+1}} K^m(c_1, y) \right| |x - c'|^{N+1} \, dx,
\]
where \( c' - h < x < c_1 < c' \). We are supposing \( y \geq 1 \) and so by Lemma 2, (8) and (9) we have
\[
|G^m(a; y)| \leq C \int_{c' - h}^{c'} |a(x)||y^{N+1}| |x - c'|^{N+1} \, dx
\]
\[
\leq C' y^\lambda \|a\|_2^{1 + \frac{2p}{1 - p}(\lambda + 1/2)}, \quad \lambda = N + 1,
\]
where \( C \) and \( C' \) are constants independent of \( a \) and \( y \). The last inequality follows from the following small lemma which will be given for later convenience, and three more simple lemmas will be also stated here.

**Lemma 4.** Let \( a \) be an \( H^p \)-atom with the support interval \( [c, c + h] \subset [0, \infty) \). Let \( \lambda > 0 \). Then the following inequality holds:
\[
\int_0^\infty |a(x)|(|y|x - c'|)^\lambda \, dx \leq y^\lambda \|a\|_2^{1 + \frac{2p}{1 - p}(\lambda + 1/2)},
\]
where \( c' \) is an arbitrary point with \( c \leq c' \leq c + h \).

**Proof.** It follows from \( \|a\|_2 \leq h^{-1/p+1/2} \), that is, \( h \leq \|a\|_2^{2p/(2-p)} \) that
\[
\int_0^\infty |a(x)|(|y|x - c'|)^\lambda \, dx \leq y^\lambda \|a\|_2 \left( \int_c^{c+h} |x - c'|^{2\lambda} \, dx \right)^{1/2}
\]
\[
\leq y^\lambda \|a\|_2 \lambda^{1/2} h^{\lambda + 1/2} \leq y^\lambda \|a\|_2^{1 + \frac{2p}{1 - p}(\lambda + 1/2)}.
\]

**Lemma 5.** Let \( 0 < p \leq 1 \). Then for an arbitrary \( \lambda \) with \( 1/p - 1 < \lambda \) and any \( a \in L^2[0, \infty) \),
\[
\int_0^R y^\lambda \|a\|_2^{1 + \frac{2p}{1 - p}(\lambda + 1/2)} \, y^{p-2} \, dy = \frac{1}{p(\lambda + 1) - 1},
\]
where \( R \) satisfies
\[
\|a\|_2^p R^{-(2-p)/2} = 1.
\]
Proof. It follows that
\[
\int_R^\infty \left( y^\lambda \|a\|_2^{1+\frac{2p}{2}(\lambda+1/2)} \right)^p y^{p-2} dy = \|a\|_2^{p(1+\frac{2p}{2}(\lambda+1/2))} \int_R^\infty y^{p(\lambda+1)-2} dy
\]

\[
= \frac{1}{p(\lambda+1)-1} \|a\|_2^{p(1+\frac{2p}{2}(\lambda+1/2))} R^{p(\lambda+1)-1}
\]

\[
= \frac{1}{p(\lambda+1)-1} \left\{ \|a\|_2^{p(\lambda+1)-1/(1+\frac{2p}{2}(\lambda+1/2))} \right\}^{1+\frac{2p}{2}(\lambda+1/2)}
\]

\[
= \frac{1}{p(\lambda+1)-1}.
\]

Here, we used the fact that the power to the last \( R \) is equal to \(-(2-p)/2\). \( \square \)

Lemma 6. Let \( 0 < p \leq 1 \) and \(-m + 1/2 > 0\). Then for any \( a \in L^2[0, \infty) \) and a constant \( R \) satisfying (18),

\[
\int_R^\infty |G^m(a; y)|^p y^{p-2} dy \leq 1, \quad \int_R^\infty |H^m(x)|^p x^{p-2} dy \leq 1.
\]

Proof. By Plancherel’s theorem, we have that

\[
\int_R^\infty |G^m(a; y)|^p y^{p-2} dy \leq \left( \int_R^\infty |G^m(a; y)|^2 dy \right)^{p/2} \left( \int_R^\infty y^{-2} dy \right)^{(2-p)/2}
\]

\[
\leq \|a\|_2^{p(2-p)} = 1.
\]

In the same way, we have the \( H \)-transform case. \( \square \)

Lemma 7. Let \( I(x), J(x) \) be nonnegative functions on \((0, \infty)\).

(i) If \( I(x) \leq J(x) \) for \( 0 < x < 1 \), then the inequality

\[
\int_0^1 I(x) \, dx \leq \int_0^R J(x) \, dx + \int_R^\infty I(x) \, dx
\]

holds for every \( R > 0 \).

(ii) If \( I(x) \leq J(x) \) for \( 1 \leq x \), then the inequality

\[
\int_1^\infty I(x) \, dx \leq \int_0^R J(x) \, dx + \int_R^\infty I(x) \, dx
\]

holds for every \( R > 0 \).

We go back to the proof. By (17) and Lemma 7, we have that for every \( R > 0 \),

\[
\int_1^\infty |G^m(a; y)|^p y^{p-2} dy \leq \int_0^R \left( C^\lambda \|a\|_2^{1+\frac{2p}{2}(\lambda+1/2)} \right)^p y^{p-2} dy
\]

\[+ \int_R^\infty |G^m(a; y)|^p y^{p-2} dy.
\]

Taking \( R \) with (18), we have by Lemma 5 and Lemma 6 that

\[
\int_1^\infty |G^m(a; y)|^p y^{p-2} dy \leq C,
\]

where \( C \) is a constant independent of \( a \). Here, we need the condition

\[
1/p - 1 < \lambda = N + 1 = [1/p],
\]

and it is trivially satisfied. This completes the proof of Theorem 1 (i).
Proof of Theorem 1 (ii). Let $0 < p \leq 1$ and $-m + 1/2 > 0$. In the same way as the above, we have (16). Now we are dealing with the case $0 < y < 1$, and our assumption is that $N + 1 \leq M = [-m + 1/2]$. Thus by the estimates (2) and (5), we have (17) for $0 < y < 1$. It follows from Lemma 7 that
\[
\int_{0}^{1} |G^m(a; y)|^p y^{p-2} \, dy \leq \int_{0}^{R} \left( C \rho^{\lambda} \|a\|_2^{1+\frac{2p}{p}} (\lambda+1/2) \right)^p y^{p-2} \, dy + \int_{R}^{\infty} |G^m(a; y)|^p y^{p-2} \, dy,
\]
and taking $R$ with (18), by Lemma 5 and 6 we have
\[
\int_{0}^{1} |G^m(a; y)|^p y^{p-2} \, dy \leq C,
\]
where $C$ is a constant independent of $a$. The condition $1/p - 1 < N + 1 = [1/p]$ is automatically satisfied.

Proof of Theorem 2 (i). Let $0 < p \leq 1$ and $-m + 1/2 > 0$, and put $N = [1/p] - 1$, $M = [-m + 1/2]$. We divide a matter into two cases $N + 1 \leq M$ and $M < N + 1$.

Let us deal with the case $N + 1 \leq M$. Let $a$ be an $H^p$-atom with the support interval $[c-h, c] (\subset [0, \infty))$. We first suppose that $c-h < 1 < c$. By the vanishing mean property of atoms, we have that
\[
|H^m(a; x)| \leq \int_{c-h}^{c} |a(y)| \left| \frac{\partial^{N+1}}{\partial y^{N+1}} K^m(x, c_2) \right| |y-1|^{N+1} \, dy
\]
\[
= \left\{ \int_{c-h}^{1} + \int_{1}^{c} \right\} |a(y)| \left| \frac{\partial^{N+1}}{\partial y^{N+1}} K^m(x, c_2) \right| |y-1|^{N+1} \, dy
\]
\[
= J_1(x) + J_2(x), \quad \text{say},
\]
where $c-h < y < c_2 < 1$ or $1 < c_2 < y < c$. We are now treating the case $1 \leq x$. It follows from Lemma 2 (9) and Lemma 4 that
\[
J_2(x) \leq C \int_{1}^{c} |a(y)| (|x| y-1)^{N+1} \, dy \leq C x^\lambda \|a\|_2^{1+\frac{2p}{p}} (\lambda+1/2), \quad \lambda = N + 1,
\]
where $C$ is independent of $x$ and $a$. For $J_1(x)$, since $N + 1 \leq M$, Lemma 2 (10) with $j = N + 1$ leads to
\[
J_1(x) \leq C_1 \int_{c-h}^{1} |a(y)| \left| \frac{\partial^{N+1}}{\partial y^{N+1}} \{ (xy)^{1/2} J_m(xy) \} \right|_{y=c_2} |y-1|^{N+1} \, dy
\]
\[
+ C_2 \int_{c-h}^{1} |a(y)||x||y-1|^{N+1} \, dy = C_1 J_{10}(x) + C_2 J_{11}(x), \quad \text{say},
\]
and $J_{11}(x) \leq x^\lambda \|a\|_2^{1+\frac{2p}{p}} (\lambda+1/2), \lambda = N + 1$, where $C_1$ and $C_2$ are independent of $x$ and $a$. For the term $J_{10}(x)$, by using the estimate
\[
\sup_{t>0} \left| \frac{\partial^j}{\partial t^j} J_\alpha(t) \right| < \infty, \quad j = 0, 1, 2, \ldots, [\alpha + 1/2], \quad \alpha \geq -1/2
\]
([11, Lemma 1, (8)]), we have that
\[
J_{10}(x) \leq C \int_{c-h}^{1} |a(y)||x||y-1|^{N+1} \, dy \leq C x^\lambda \|a\|_2^{1+\frac{2p}{p}} (\lambda+1/2), \quad \lambda = N + 1,
\]
Lemma 4 that the assumption 1

\begin{equation}
C \leq a \leq \frac{2}{(\lambda + 1/2)^2}, \quad \lambda = N + 1, \quad 1 \leq x
\end{equation}

with a constant \( C \) independent of \( x \) and \( a \) for an \( H^p \)-atom \( a \) with the support interval \( [c-h,c] \) satisfying \( c-h < 1 < c \). For the case \( 1 \leq c-h \), we also have the above estimate (20) in the same way as the argument for \( J_2(x) \), and for the case \( c \leq 1 \), we have (20) in the same way as the argument for \( J_1(x) \). Lemma 7 leads to

\begin{align*}
\int_1^\infty |\mathcal{H}^m(a;x)|^p x^{p-2} \, dx & \leq \int_0^R \left( C x^\lambda \|a\|_2^{1 + \frac{22}{(\lambda + 1/2)^2}} \right)^p x^{p-2} \, dx \\
& \quad + \int_R^\infty |\mathcal{H}^m(a;x)|^p x^{p-2} \, dx, \quad \lambda = N + 1
\end{align*}

for any \( R > 0 \) and every \( H^p \)-atom \( a \) with the support interval contained in \([0, \infty)\). Noting \( 1/p - 1 < \lambda \) and taking \( R \) with (18), we have by Lemma 5 and Lemma 6 that

\begin{equation}
\int_1^\infty |\mathcal{H}^m(a;x)|^p x^{p-2} \, dx \leq C, \quad N + 1 \leq M
\end{equation}

with a constant \( C \) independent of \( a \).

Next we treat the case \( M < N + 1 \). We first examine the case \(-m+1/2 = 1, 2, 3, \ldots\). Because of (9) and (11), we have by the vanishing mean properties and Lemma 4 that

\begin{align*}
|\mathcal{H}^m(a;x)| & \leq \int_{c-h}^c |a(y)| \left| \frac{\partial^{N+1}}{\partial y^{N+1}} K^m(x, c_2) \right| |y - c|^{N+1} \, dy \\
& \leq \int_{c-h}^c |a(y)|(y-c)^{N+1} \, dy \leq x^\lambda \|a\|_2^{1 + \frac{22}{(\lambda + 1/2)^2}}, \quad \lambda = N + 1,
\end{align*}

where \( c-h < y < c_2 < c \) and \( a \) is an \( H^p \)-atom with the support interval \([c-h,c] \subseteq [0, \infty)\). In the same way as the above argument, we have

\begin{equation}
\int_1^\infty |\mathcal{H}^m(a;x)|^p x^{p-2} \, dx \leq C, \quad M < N + 1, \quad -m+1/2 = 1, 2, 3, \ldots,
\end{equation}

where \( C \) is independent of \( a \).

Let us consider the case \(-m+1/2 \neq 1, 2, 3, \ldots\). In this case we suppose that \( 1/p - 1 < -m + 1/2 \). Since \( M < N + 1 \), it follows that \(-m+1/2 < N + 1\). By the assumption \( 1/p - 1 < -m + 1/2 \), we have \( N < -m + 1/2 \). Thus, in this case, \( N < -m + 1/2 < N + 1 \) and \( M = N \) hold. Let \( a \) be an \( H^p \)-atom with the support interval \([c-h,c] \subseteq [0, \infty)\). We first deal with the case \( c-h < 1 < c \). We have that

\[ \mathcal{H}^m(a;x) = \int_{c-h}^c a(y) \left( \frac{\partial^M K^m}{\partial y^M} (x, \xi) - \frac{\partial^M K^m}{\partial y^M} (x, 1) \right) (y-1)^M \, dy, \]

and that

\[ |\mathcal{H}^m(a;x)| \leq \left\{ \int_{c-h}^1 + \int_1^c \right\} |a(y)| \left| \frac{\partial^M K^m}{\partial y^M} (x, \xi) - \frac{\partial^M K^m}{\partial y^M} (x, 1) \right| (y-1)^M \, dy \\
= J_3(x) + J_4(x), \quad \text{say,} \]

where \( c-h < y < \xi < 1 \) or \( 1 < \xi < y < c \). Since \( M = N \), it follows that

\[ J_4(x) = \int_1^c |a(y)| \left| \frac{\partial^{N+1} K^m}{\partial y^{N+1}} (x, \xi') \right| (y-1)^{N+1} \, dy, \quad 1 < \xi' < y < c. \]
We are now dealing with the case $1 \leq x$. By Lemma 2 (9), we have that
\[ J_4(x) \leq C \int_1^c |a(y)|(x|y - 1|)^{N+1} \, dy \]
with a constant $C$ independent of $x$ and $a$, and by Lemma 4 that
\[ J_4(x) \leq C x^\lambda \|a\|^2_2 + \frac{2^p}{2^p}(\lambda + 1/2), \quad \lambda = N + 1. \]
For $J_3(x)$, it follows from Lemma 2 (10) that
\[
J_3(x) \leq C_1 \int_{c-h}^1 |a(y)| \left| \frac{\partial M}{\partial y} \{(xy)^{1/2}J_{-m}(xy)\} \right|_{y=x} \\
- \frac{\partial M}{\partial y} \{(xy)^{1/2}J_{-m}(xy)\} \left|_{y=1} \right| |y - 1|^M \, dy \\
+ C_2 \int_{c-h}^1 |a(y)| \left| \frac{\partial M+1}{\partial y} E_m \{(x, \xi)\} \right| |y - 1|^{M+1} \, dy \\
= C_1 J_{30}(x) + C_2 J_{31}(x), \quad \text{say},
\]
where $C_1$ and $C_2$ are independent of $x$ and $a$. Since $M = N$, it follows from Lemma 4 that
\[ J_{31}(x) \leq C \int_{c-h}^1 |a(y)|(x|y - 1|)^{N+1} \, dy \leq C x^\lambda \|a\|^2_2 + \frac{2^p}{2^p}(\lambda + 1/2), \quad \lambda = N + 1 \]
with a constant $C$ independent of $x$ and $a$. By using the estimate [11, Lemma 1, (9)], we have
\[
\left| \frac{\partial M}{\partial y} \{(xy)^{1/2}J_{-m}(xy)\} \right|_{y=x} \\
- \frac{\partial M}{\partial y} \{(xy)^{1/2}J_{-m}(xy)\} \left|_{y=1} \right| \\
\leq C x^M |x \xi - x|^{-m+1/2-M},
\]
where $c - h < y < \xi < 1$ and $C$ is independent of $x$. Thus it follows from Lemma 4 that
\[ J_{30}(x) \leq C \int_{c-h}^1 |a(y)|(x|y-1|)^{-m+1/2} \, dy \leq C x^{\lambda'} \|a\|^2_2 + \frac{2^p}{2^p}(\lambda' + 1/2), \quad \lambda' = -m+1/2 \]
with a constant $C$ independent of $x$ and $a$. Thus for an $H^p$-atom $a$ with the support interval $[c-h, c]$ satisfying $c - h < 1 < c$ we have
\[
|\mathcal{H}^m(a; x)| \leq C_1 x^{\lambda'} \|a\|^2_2 + C_2 x^\lambda \|a\|^2_2 + + \frac{2^p}{2^p}(\lambda + 1/2), \\
\lambda = N + 1, \quad \lambda' = -m+1/2, \quad 1 \leq x
\]
with constants $C_1$ and $C_2$ independent of $x$ and $a$. For the case $1 \leq c - h$, we make the same argument for $J_4(x)$, and have
\[
|\mathcal{H}^m(a; x)| \leq C x^{\lambda} \|a\|^2_2 + \frac{2^p}{2^p}(\lambda + 1/2), \quad \lambda = N + 1, \quad 1 \leq x
\]
For the case $c \leq 1$, the same argument for $J_3(x)$ leads to
\[
|\mathcal{H}^m(a; x)| \leq C x^\lambda \|a\|^2_2 + \frac{2^p}{2^p}(\lambda' + 1/2), \quad \lambda = -m+1/2, \quad 1 \leq x.
\]
Therefore for any atoms we have (23). It follows that for every \( R > 0 \),
\[
\int_1^\infty |\mathcal{H}(a; x)|^p x^{p-2} \, dx \leq \int_0^R \left( C_1 x^\lambda \|a\|_2^1 \frac{2^p}{2} (\lambda + 1/2) \right)^p x^{p-2} \, dx \\
+ \int_0^R \left( C_2 x^\lambda \|a\|_2^1 \frac{2^p}{2} (\lambda + 1/2) \right)^p x^{p-2} \, dx \\
+ \int_0^\infty |\mathcal{H}(a; x)|^p x^{p-2} \, dx, \quad \lambda = N + 1, \lambda' = -m + 1/2.
\]
Taking \( R \) with (18) and noting \( 1/p - 1 < \lambda (= N + 1 = [1/p]) \) and \( 1/p - 1 < -m + 1/2 \),
we have by Lemma 5 and Lemma 6 that
\[
\int_1^\infty |\mathcal{H}(a; x)|^p x^{p-2} \, dx \leq C,
\]
with a constant \( C \) independent of \( a \). The inequalities (21), (22) and (24) complete
the proof of Theorem 2 (i).

Proof of Theorem 2 (ii). Assume that \( -m + 1/2 > 0 \) and \( 1/2 < p \leq 1 \). It is
clear that \( N = \lfloor 1/p \rfloor - 1 = 0 \). Let \( a \) be an \( H^p \)-atom with the support
interval \([c - h, c] \subset (0, \infty)\).

We treat the case \( c - h < 1 < c \), first. Noting that
\[
\mathcal{H}(a; x) = \int_{c-h}^c a(y)(K^m(x, y) - K^m(x, 1)) \, dy,
\]
we have
\[
|\mathcal{H}(a; x)| \leq \int_{c-h}^c |a(y)||K^m(x, y) - K^m(x, 1)| \, dy \\
= \left\{ \int_{c-h}^1 + \int_1^c \right\} |a(y)||K^m(x, y) - K^m(x, 1)| \, dy \\
= J_6(x) + J_5(x), \quad \text{say.}
\]
We are now supposing that \( 0 < x < 1 \). For \( J_6(x) \), it follows from Lemma 2 (8) and
Lemma 4 that
\[
J_6(x) = \int_1^c |a(y)||K^m(x, y) - K^m(x, 1)| \, dy = \int_1^c |a(y)| \left| \frac{\partial K^m}{\partial y} (x, \xi) \right| y - 1 \, dy \\
\leq C \int_1^c |a(y)|(y|y - 1|) \, dy \leq C x^\lambda \|a\|_2^{1 + \frac{2p}{p} (\lambda + 1/2)}, \quad \lambda = 1,
\]
where \( 1 < \xi < y < c \) and \( C \) is independent of \( x \) and \( a \). For \( J_5(x) \), we divide a
matter into two cases \( M = [-m + 1/2] = 0 \) and \( M \geq 1 \). Let \( M \geq 1 \). Because of
Lemma 2 (6), the same argument for \( J_6(x) \) leads to
\[
J_5(x) = \int_{c-h}^1 |a(y)||K^m(x, y) - K^m(x, 1)| \, dy \\
= \int_{c-h}^1 |a(y)| \left| \frac{\partial K^m}{\partial y} (x, \xi) \right| y - 1 \, dy \\
\leq C \int_{c-h}^1 |a(y)|(y|y - 1|) \, dy \leq C x^\lambda \|a\|_2^{1 + \frac{2p}{p} (\lambda + 1/2)}, \quad \lambda = 1.
\]
Hardy-type inequalities for the generalized Mehler transform

We next deal with the case $M = 0$. We remark

$0 < -m + 1/2 < 1$. It follows fromLemma 2 (7) and Lemma 4 that

$$J_5(x) = \int_{c-h}^{1} |a(y)||K^m(x, y) - K^m(x, 1)| \, dy = \int_{c-h}^{1} |a(y)| |x| \, dy$$

$$\leq C \int_{c-h}^{1} |a(y)|(x|y - 1|) \, dy \leq C x^\lambda \|a\|_2^{1 + \frac{2p}{p-1}(\lambda + 1/2)}, \quad \lambda = -m + 1/2.$$

We used that $x < x^\delta$ $(0 < x < 1)$ since $1 > \delta = -m + 1/2 - M = -m + 1/2 > 0$. Thus for an $H^p$-atom $a$ with the support interval $[c - h, c]$ satisfying $c - h < 1 < c$ we have

$$|H^m(a; x)| \leq C_1 x^{\lambda'} \|a\|_2^{1 + \frac{2p}{p-1}(\lambda' + 1/2)} + C_2 x^\lambda \|a\|_2^{1 + \frac{2p}{p-1}(\lambda + 1/2)}, \quad \lambda = 1, \lambda' = -m + 1/2, \quad 0 < x < 1$$

with constants $C_1$ and $C_2$ independent of $x$ and $a$.

For the case $1 \leq c - h$, by the same argument for $J_6(x)$ we have

$$|H^m(a; x)| \leq C x^\lambda \|a\|_2^{1 + \frac{2p}{p-1}(\lambda + 1/2)}, \quad \lambda = 1, \quad 0 < x < 1$$

with a constant $C$ independent of $x$ and $a$. For the case $c \leq 1$, in a similar way of the argument for $J_5(x)$ we have (25). Therefore we have (25) for any atom. It follows from Lemma 7 that for every $R > 0$,

$$\int_0^1 |H^m(a; x)|^p x^{p-2} \, dx$$

$$\leq \int_0^R \left( C_1 x^{\lambda'} \|a\|_2^{1 + \frac{2p}{p-1}(\lambda' + 1/2)} \right)^p x^{p-2} \, dx + \int_0^R \left( C_2 x^\lambda \|a\|_2^{1 + \frac{2p}{p-1}(\lambda + 1/2)} \right)^p x^{p-2} \, dx$$

$$+ \int_R^\infty |H^m(a; x)|^p x^{p-2} \, dx, \quad \lambda = 1, \lambda' = -m + 1/2.$$

We take $R$ as it satisfies (18). Noting that $1/p - 1 < -m + 1/2$ and $1/p - 1 < 1$, we have by Lemma 5 and Lemma 6 that

$$\int_0^1 |H^m(a; x)|^p x^{p-2} \, dx \leq C$$

with a constant $C$ independent of $a$, which completes the proof of Theorem 2 (ii), and the proofs of the theorems complete.
References

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