A note on almost contact Riemannian 3-manifolds

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Abstract

We investigate curvatures of normal almost contact Riemannian 3-manifolds. In particular, we show that Kenmotsu 3-manifolds of constant scalar curvature are of constant curvature \(-1\).

Introduction

In [6], K. Kenmotsu introduced a class of almost contact Riemannian manifolds. The almost contact Riemannian manifolds introduced by Kenmotsu are called Kenmotsu manifolds. Kenmotsu showed that locally symmetric Kenmotsu manifolds are of constant curvature \(-1\). This fact means that local symmetry is a strong restriction for Kenmotsu manifolds.

In stead of local symmetry, U. C. De [4] studied Kenmotsu manifolds \(M = (M; \varphi, \xi, \eta, g)\) satisfying

\[ \varphi^2 \{(\nabla_W R)(X, Y) Z\} = 0 \]

for all \(X, Y, Z, W \in \mathfrak{X}(M)\) orthogonal to \(\xi\). He showed that if \(M\) satisfies (1) for all vector fields on \(M\), then \(M\) is Einstein. In dimension 3, De showed that a Kenmotsu 3-manifold \(M\) satisfies (1) for all vector fields orthogonal to \(\xi\) if and only if \(M\) is of constant scalar curvature.

In this paper we point out that Kenmotsu 3-manifolds of constant scalar curvature are of constant curvature \(-1\). Thus De’s condition on Kenmotsu 3-manifolds implies local symmetry.

1 Preliminaries

Let \((M, g)\) be a Riemannian manifold with Levi-Civita connection \(\nabla\). Denote by \(R\) the Riemannian curvature of \(M\):

\[ R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \quad X, Y \in \mathfrak{X}(M). \]

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Here $\mathfrak{X}(M)$ is the Lie algebra of all vector fields on $M$. A tensor field $F$ of type $(1,3)$;

$$F : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$$

is said to be curvature-like provided that $F$ has the symmetric properties of $R$.

For example,

$$(X \wedge Y)Z = g(Y, Z)X - g(Z, X)Y, \quad X, Y \in \mathfrak{X}(M)$$

defines a curvature-like tensor field on $M$. Note that the curvature $R$ of a Riemannian manifold $(M, g)$ of constant curvature $c$ satisfies the formula $R(X, Y) = c(X \wedge Y)$.

A Riemannian manifold $(M, g)$ is said to be locally symmetric if $\nabla R = 0$. Clearly every Riemannian manifold of constant curvature is locally symmetric.

In dimension 3, the Riemannian curvature $R$ is determined by the Ricci tensor. In fact, $R$ is expressed as

$$R(X, Y)Z = \rho(Y, Z)X - \rho(Z, X)Y + g(Y, Z)SX - g(Z, X)SY - \frac{s}{2}(X \wedge Y)Z,$$

where $\rho$ is the Ricci tensor, $S$ is the corresponding Ricci operator and $s$ is the scalar curvature of $M$, respectively.

## 2 Almost contact Riemannian manifolds

Let $M$ be an odd-dimensional manifold. An almost contact structure on $M$ is a quadruple of tensor fields $(\varphi, \xi, \eta, g)$, where $\varphi$ is an endomorphism field, $\xi$ is a vector field, $\eta$ is a one form and $g$ is a Riemannian metric, respectively, such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1,$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad X, Y \in \mathfrak{X}(M).$$

An $(2n + 1)$-dimensional manifold together with an almost contact structure is called an almost contact Riemannian manifold (or almost contact manifold). The fundamental 2-form $\Phi$ of $M$ is defined by

$$\Phi(X, Y) = g(X, \varphi Y), \quad X, Y \in \mathfrak{X}(M).$$

If an almost contact Riemannian manifold $(M; \varphi, \xi, \eta, g)$ satisfies the condition:

$$\rho = ag + b\eta \otimes \eta$$

for some functions $a$ and $b$, then $M$ is said to be $\eta$-Einstein.

The formulae (3) and (6) imply the following result.
Proposition 2.1 Let $M$ be an $\eta$-Einstein almost contact Riemannian 3-manifold. Then its Riemannian curvature $R$ is given by

$$R(X,Y)Z = \left(2a - s\right)(X \wedge Y)Z - [(b\xi) \wedge \{(X \wedge Y)\xi\}]Z.$$  

An almost contact Riemannian manifold $M$ is said to be normal if it satisfies $[\varphi,\varphi] + 2d\eta \otimes \xi = 0$, where $[\varphi,\varphi]$ is the Nijenhuis torsion of $\varphi$.

Proposition 2.2 ([7]) An almost contact Riemannian 3-manifold is normal if and only if there exist functions $\alpha$ and $\beta$ such that

$$\nabla_X \varphi Y = \alpha \{g(X,Y)\xi - \eta(Y)X\} + \beta \{g(\varphi X,Y)\xi - \eta(Y)\varphi X\}.$$  

We call the pair $(\alpha, \beta)$ of functions the type of a normal almost contact Riemannian 3-manifold $M$. More generally, an almost contact manifold of dimension $2n + 1 \geq 3$ is said to be trans-Sasakian if there exist functions $\alpha$ and $\beta$ such that (8) (see [9]).

In particular, a normal almost contact Riemannian 3-manifold is said to be a

- Sasakian manifold if $(\alpha, \beta) = (1, 0),$
- Kenmotsu manifold if $(\alpha, \beta) = (0, 1),$
- coKähler manifold if $(\alpha, \beta) = (0, 0).$

Let $(M; \varphi, \xi, \eta, g)$ be a normal almost contact Riemannian 3-manifold. Then from (4) and (8), we have

$$\nabla_X \xi = -\alpha \varphi X + \beta \{X - \eta(X)\xi\}, \ X, Y \in \mathfrak{X}(M).$$

In particular we have $\nabla_\xi \xi = 0$. Hence on trans-Sasakian manifolds, integral curves (trajectories) of $\xi$ are geodesics.

Next, we consider $\eta$-Einstein normal almost contact Riemannian 3-manifolds.

Proposition 2.3 ([3]) Let $M$ be a normal almost contact Riemannian 3-manifold of type $(\alpha, \beta)$. Then $M$ is $\eta$-Einstein if and only if

$$g(\nabla\varphi - \varphi\nabla \alpha, X) = 0$$

for all $X \in \mathfrak{X}(M)$ orthogonal to $\xi$. In this case,

$$\rho = \left\{\frac{s}{2} + d\beta(\xi) - (\alpha^2 - \beta^2)\right\}g + \left\{-\frac{s}{2} - 3d\beta(\xi) + 3(\alpha^2 - \beta^2)\right\}\eta \otimes \eta.$$ 

Corollary 2.1 The Riemannian curvature of a Sasakian 3-manifold is given by

$$R(X,Y)Z = \frac{s - 4}{2}(X \wedge Y)Z + \frac{s - 6}{2}[\xi \wedge \{(X \wedge Y)\xi\}]Z.$$  

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Corollary 2.2 The Riemannian curvature of a Kenmotsu 3-manifold is given by
\[ R(X,Y)Z = \frac{s + 4}{2} (X \wedge Y)Z + \frac{s + 6}{2} [\xi \wedge ((X \wedge Y)\xi)]Z. \]

Corollary 2.3 The Riemannian curvature of a coKähler 3-manifold is given by
\[ R(X,Y)Z = \frac{s}{2} [\xi \wedge ((X \wedge Y)\xi)]Z. \]

3 Kenmotsu 3-manifolds

Let \((N,h,J)\) be a Riemannian 2-manifold together with the compatible orthogonal complex structure \(J\). Take a direct product \(M = E^1(t) \times N\) of real line \(E^1(t)\) and \(N\). We denote \(\pi\) and \(\sigma\) the natural projections onto the first and second factors,
\[ \pi : M \to E^1, \quad \sigma : M \to N, \]
respectively. On the direct product \(M\), we equip a Riemannian metric \(g\) defined by
\[ g = dt^2 + f(t)^2 \pi^* h. \]
Here \(f\) is a positive function on \(E^1(t)\). The resulting Riemannian manifold \((M,g)\) is denoted by \(E^1 \times_f N\) and called the warped product with base \(E^1\) and fibre \(N\). The function \(f\) is called the warping function.

On the warped product \(M = E^1 \times_f N\), we define the vector field \(\xi\) by \(\xi = \frac{\partial}{\partial t}\). Then the Levi-Civita connection \(\nabla\) of \(M\) is given by (cf. [8]):
\[ \nabla_X Y^v = (\nabla_X Y)^v - \frac{1}{f} g(X^v,Y^v) f' \xi, \]
\[ \nabla_X \xi^v = \nabla_X^v \xi = \frac{f'}{f} X^v, \]
\[ \nabla_\xi \xi = 0. \]
Here the superscript \(v\) means the vertical lift operation of vector fields from \(N\) to \(M\). Define \(\varphi\) by \(\varphi X = (J(\sigma* X))^v\). Then we get
\[ \nabla_X \xi = \beta(X - \eta(X)\xi), \]
\[ (\nabla_X \varphi)Y = \beta\{g(\varphi X,Y) - \eta(Y)\varphi X\}, \quad \beta = f'/f. \]
Hence \(M = E^1 \times_f N\) is a normal almost contact Riemannian 3-manifold of type \((0,\beta)\). In particular \(E^1 \times_f N\) is a Kenmotsu manifold if and only if \(f(t) = ce^t\) for some positive constant \(c\). Take a local orthonormal frame field \(\{\bar{e}_1, \bar{e}_2\}\) of \((N,h)\) such that \(\bar{e}_2 = J\bar{e}_1\). Then we obtain a local orthonormal frame field \(\{e_1,e_2,e_3\}\) by
\[ e_1 = \frac{1}{f} \bar{e}_1, \quad e_2 = \frac{1}{f} \bar{e}_2 = \varphi e_1, \quad e_3 = \xi. \]
Then sectional curvatures of $M$ are given by
\[ K(e_1 \wedge e_2) = \frac{1}{f^2} \{ \kappa - (f')^2 \}, \quad K(e_1 \wedge e_3) = K(e_2 \wedge e_3) = -\frac{f''}{f}, \]
where $\kappa$ is the Gaussian curvature of $N$. The Ricci tensor components $\rho_{ij} = \rho(e_i, e_j)$ are given by
\[ \rho_{11} = \rho_{22} = \frac{\kappa}{f^2} - \frac{f''}{f} - \left( \frac{f'}{f} \right)^2, \quad \rho_{33} = \frac{2f''}{f}. \]

The local structure of Kenmotsu manifolds is described as follows.

**Lemma 3.1** ([6]) A Kenmotsu 3-manifold $M$ is locally isomorphic to a warped product $I \times_f N$ whose base $I \subset \mathbb{E}^1(t)$ is an open interval, $N$ is a surface and warping function $f(t) = ce^t$, $c > 0$. The structure vector field is $\xi = \partial/\partial t$.

**Proposition 3.1** A Kenmotsu 3-manifold is of constant scalar curvature if and only if $M$ is of constant curvature $-1$.

(Proof.) For every point $p \in M$, there exists a neighbourhood $U_p$ of $p$ such that $U_p$ is a warped product $(-\epsilon, \epsilon) \times_f N$ of an open interval $(-\epsilon, \epsilon)$ and a Riemannian 2-manifold of Gaussian curvature $\kappa$ with warping function $f(t) = ce^t$. The scalar curvature $s$ over $U_p$ is computed as
\[ s|_{U_p} = -6 + 2\kappa c^{-2}e^{-2t}. \]
Thus the differential $ds$ is computed as
\[ \frac{1}{2} ds = c^{-2}e^{-2t}d\kappa - 2\kappa c^{-2}e^{-2t}dt. \]
Hence $ds = 0$ if and only if $\kappa = 0$. This implies that $U_p$ is of constant curvature $-1$. \(\blacksquare\)

**Corollary 3.1** A Kenmotsu 3-manifold satisfies the condition (1) for all $X, Y, Z, W \in \mathfrak{X}(M)$ orthogonal to $\xi$ if and only if $M$ is locally symmetric.

(Proof.) De [4] showed that $M$ satisfies (1) for all $X, Y, Z, W \in \mathfrak{X}(M)$ orthogonal to $\xi$ if and only if $M$ is of constant scalar curvature. As we have seen above, $M$ is of constant scalar curvature if and only if $M$ is of constant curvature $-1$. \(\blacksquare\)

Note that all the examples of Kenmotsu 3-manifold exhibited in [4, Example 5.1, 5.2, 5.3] are of constant curvature $-1$. 

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References


