Locally Conformal Almost Cosymplectic Manifolds Endowed with a Skew-Symmetric Killing Vector Field

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Abstract

We study a locally conformal almost cosymplectic manifold $M$ which carries a horizontal skew-symmetric Killing vector field $X$. Such $X$ defines a relative conformal cosymplectic transformation of the conformal cosymplectic 2-form $\Omega$ of $M$ and the square of its length is both an isoparametric function and an eigenfunction of the Laplacian.

Keywords: Locally conformal almost cosymplectic manifolds, skew-symmetric Killing vector field, infinitesimal concircular transformation, relative conformal cosymplectic transformation.

1 Preliminaries

Let $(M,g)$ be an oriented $n$-dimensional Riemannian $C^\infty$-manifold and $\nabla$ be the covariant differential operator with respect to the metric tensor $g$. Let $\Gamma TM$ be the set of sections of the tangent bundle and $\flat : TM \rightarrow T^*M$ and $\flat^\dagger = \flat^{-1}$ the classical musical isomorphisms defined by $g$. We denote by $A^q(M,TM)$ the set of all vector valued $q$-forms, $q < \dim M$.

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A vector field $U$ is said to be \textit{exterior concurrent} if it satisfies
\begin{equation}
\label{eq:1.1}
\nabla^2 U = \alpha \wedge dp \in A^2(M, TM), \quad \alpha \in \Lambda^1(M, TM),
\end{equation}
where $\alpha = \lambda U^\flat$ for a certain $\lambda \in \Lambda^0$ and it is called a \textit{concurrence form} ([MRV], [PRV], [R2]).

In (1.1), $\alpha$ is called the \textit{concurrence form} and is defined by
\[ \alpha = \lambda U^\flat, \quad \lambda \in \Lambda^0 M. \]

A function $f : M \rightarrow \mathbb{R}$ is \textit{isoparametric} if $\| \nabla f \|$ and $\text{div}(\nabla f)$ are functions of $f$ ([W]).

Let $\mathcal{O} = \{ e_A | A = 1, \ldots, n \}$ be a local field of orthonormal frame over $M$ and let $\mathcal{O}^* = \{ \omega^A \}$ be its associated coframe. Then the soldering form $dp$ is expressed by $dp = \omega \otimes e$. Also, the Cartan’s structure equations written in indexless manner are
\begin{align}
\nabla e &= \theta \otimes e, \label{eq:1.2} \\
d\omega &= -\theta \wedge \omega, \label{eq:1.3} \\
d\theta &= -\theta \wedge \theta + \Theta. \label{eq:1.4}
\end{align}

In the above equations, $\theta$ (resp. $\Theta$) are the \textit{local connection forms} in the tangent bundle $TM$ (resp. the \textit{curvature forms} on $M$).

A $(2m + 1)$-dimensional \textit{locally conformal almost cosymplectic manifold} $M$ with structure $(\phi, \Omega, \xi, \eta, g)$ is defined by
\[ d\Omega = 2\omega \wedge \Omega, \quad \eta = \omega \wedge \eta, \]
for certain 1-form $\omega$, where $\phi$ is an endomorphism of the tangent bundle $TM$ of square $-1$, $\Omega$ is the structure 2-form, which is called a \textit{locally conformal almost cosymplectic 2-form}, $\Omega$ a conformal cosymplectic 2-form of rank $2m$, $\xi$ the Reeb vector field and $\eta$ the Reeb covector field.

It is known that the 1-form $\omega$ from the above equation is a closed 1-form which is called the \textit{characteristic form} associated with the locally conformal almost cosymplectic structure ([MMR]).

In addition, if $M$ is endowed with a quasi-Sasakian structure defined by a field $\phi$ of endomorphism of its tangent space and $\omega$ satisfies $\omega = -\eta$, then $M$ is called an \textit{almost cosymplectic $-1$-manifold}. Let $D_p^\perp$ (resp. $D_p^\perp$) be a set of all tangent vectors at $p$ which are orthogonal to (resp. proportional to) $\xi_p$. Then we may split the tangent space $T_pM$ of $M$ at $p \in M$ as $T_pM = D_p^\perp \oplus D_p^\perp.$
We can construct the distribution $D : p \rightarrow D^p \setminus \{X_p; \eta_p(X_p) = 0\}$, called the horizontal distribution and the distribution $D^\perp : p \rightarrow D^\perp_p = \{\xi_p\}$, called the vertical distribution.

In almost cosymplectic $-1$-manifold $M$, one has the following (see, for instance, [MMR], [OR])

\[(1.5) \quad d\Omega = -2\eta \land \Omega, \quad \Omega(Z, Z') = g(\phi Z, Z'),\]

\[(1.6) \quad (\nabla_{Z'}\phi)Z = \eta(Z)\phi Z' + g(\phi Z, Z')\xi,\]

\[(1.7) \quad \nabla\xi = -dp + \eta \otimes \xi,\]

\[(1.8) \quad d\eta = 0.\]

A vector field $X$ is called a horizontal skew-symmetric Killing vector field with generatives $\xi$ if it satisfies

\[(1.9) \quad \nabla X = \xi \land X, \quad \eta(X) = 0.\]

Then we have

**Lemma 1.** Let $X$ be a horizontal skew-symmetric Killing vector field. If we put $2l = ||X||^2$, then we have the following properties:

i) $2l$ is an isoparametric function,

ii) grad $2l$ defines an infinitesimal concircular transformation and

iii) $l$ is an eigenfunction of the Laplacian $\Delta$.

Also, we have

**Lemma 2.** The above vector field $X$ satisfies the following

\[\nabla^3 X = 2(X^{\flat} \land \eta) \land dp,\]

i.e., by definition, $X$ is a 2-exterior concurrent vector field, and

\[d(\mathcal{L}_X\Omega) = -2\eta \land \mathcal{L}_X\Omega,\]

i.e., by definition, $X$ defines a relative almost cosymplectic transformation of $\Omega$ ($\mathcal{L}_X\Omega$ is exterior recurrent with $-2\eta$ as recurrence form).

Proofs of the above lemmas will be given in the next section.
2 Main Result

We assume in this paper that a vector field $X$ is a skew-symmetric Killing vector field having the Reeb vector field $\xi$ as generative ([R2]), i.e.,

\begin{equation}
\nabla X = \xi \wedge X,
\end{equation}

or, equivalently,

\begin{equation}
\nabla X = X^b \otimes \xi - \eta \otimes X.
\end{equation}

Let $\mathcal{O} = \{e_A | A = 1, \ldots, 2m+1\}$ be a local field of orthonormal frame over $M$ and let $\mathcal{O}^* = \{\omega^A\}$ be its associated coframe and we assume that $e_{2m+1} = \xi$ and $\omega^{2m+1} = \eta$.

We assume that $X$ is a horizontal vector field $(\eta(X) = 0)$. Then the vector field $X$ is written as

\begin{equation}
X^a = \sum_{a=1}^{2m} X^a \omega^a
\end{equation}

and

\begin{equation}
\nabla X = (dX^a + X^b \theta^a_b) \otimes e_a + X^b \otimes \xi, \quad a, b = 1, \ldots, 2m.
\end{equation}

Hence, by (2.2), one obtains by a standard calculation

\begin{equation}
dX^a + X^b \theta^a_b = X^a \eta
\end{equation}

and setting

\begin{equation}
2l = \|X\|^2,
\end{equation}

one derives from (2.5)

\begin{equation}
dl = -2l \eta,
\end{equation}

which is concordance with (1.8). Next, from (2.7), one has $\text{grad } l = -2l \xi$, which imply

\begin{equation}
\|\text{grad } l\|^2 = 4l^2,
\end{equation}

and

\begin{equation}
\text{div(grad } l) = 4ml,
\end{equation}

108
which say that the length $2l$ of the vector field $X$ is an isoparametric function.

In addition, one has

$$g(\nabla_Z \text{grad } l, Z') = 2l g(Z, Z')$$

for any $Z, Z' \in \Gamma TM$. This means, by definition, that grad $l$ defines an *infinitesimal concircular transformation* of a vector field $Z$ ([MRV]).

In the same order of ideas, one gets

$$\Delta l = 4ml,$$

i.e., $l$ is eigenfunction of the Laplacian $\Delta$.

In this way, Lemma 1 has been proved.

Next, since $\nabla$ acts inductively, one derives

$$\nabla^2 X = X^b \wedge dp - 2(\eta \wedge X^b) \otimes \xi.$$}

This means that the distinguished vector field $X$ is a quasi-exterior concurrent vector field.

Further, one has

$$\nabla(\nabla^2 X) = \nabla^3 X = 2(X^b \wedge \eta) \wedge dp,$$

i.e., by definition, $X$ is a 2-exterior concurrent vector field ([MRV]).

Finally, regarding the conformal cosymplectic form $\Omega$, we define $\beta$

$$\beta = i_X \Omega = \sum_{a=1}^{n} (X^a \omega^a - X^a \omega^a).$$

Then, since

$$\mathcal{L}_X \Omega = d(i_X \Omega) + 2\eta \wedge i_X \Omega,$$

one may write

$$\mathcal{L}_X \Omega = d\beta + 2\eta \wedge \beta,$$

and, by exterior differentiation, one derives

$$d(\mathcal{L}_X \Omega) = -2\eta \wedge \mathcal{L}_X \Omega.$$}

Then, the relation (2.17) affirms that the distinguished vector field $X$ defines a *relative conformal cosymplectic transformation of $\Omega$* (see [R1]).
In this way, Lemma 2 has been proved.

Summing up, and making use of Lemmas 1 and 2, we proved the following.

**Theorem.** Let \( M(\phi, \Omega, \xi, \eta, g) \) be a \((2m + 1)\)-dimensional locally conformal almost cosymplectic \( C^\infty \)-manifold, with Reeb vector field \( \xi \). Then, if \( M \) carries a horizontal vector field \( X \) such that \( X \) is a skew-symmetric Killing vector field, one has the properties:

i) \( 2l = ||X||^2 \) is an isoparametric function; moreover, \( \text{grad } l \) is an infinitesimal concircular transformation and \( l \) is an eigenfunction of the Laplacian \( \Delta \);

ii) \( X \) is a closed vector field which is 2-exterior concurrent, i.e.,

\[
\nabla^3 X = 2(X^\phi \wedge \eta) \wedge dp;
\]

iii) \( X \) defines a relative conformal cosymplectic transformation of \( \Omega \), i.e.

\[
d(\mathcal{L}_X \Omega) = -2\eta \wedge \mathcal{L}_X \Omega.
\]

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**References**


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